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LETTER TO THE EDITOR

Exact solution of $Z(N)$ invariant spin models on a triangular lattice

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Abstract. Based on Baxter's ideas we present a method to solve spin models on triangular lattices. If the model has only nearest neighbour interaction and it satisfies the star–triangle relation then the exact free energy can be obtained. We apply the method to reobtain the critical triangular Potts model and to extend the self-dual $Z(N)$ solutions of Fateev and Zamolodchikov to the triangular lattice.

It has been recognised in recent years that the two-dimensional Potts model is experimentally realisable. The relevant substances are adsorbed monolayers (Alexander 1975, Bretz 1977, Tejwani *et al* 1980, Berker *et al* 1978) and silver β alumina (Gouyet *et al* 1980) and they correspond to the three-state model on triangular lattices. The interest in the triangular Potts model has since increased. From the theoretical point of view, the critical triangular Potts model was solved using an equivalence between this model and a six-vertex model on a Kagomé lattice (Baxter *et al* 1978) which is, on the other hand, equivalent to a solvable 20-vertex model (Kelland 1974). In our letter we present a method which we believe is simpler and also applies to other models where those equivalences have not been found.

In his book Baxter (1982) derived the solution of the Ising model on a triangular lattice based on the equivalence between an eight-vertex model on a Kagomé lattice and two decoupled Ising models defined on a triangular and hexagonal lattice. We will show that as long as a spin model has only nearest neighbour interactions and obeys the star–triangle relation (see Pokrowsky and Bashilov 1982) the method permits us to solve the spin model defined on a triangular lattice from the known solution of the same model on a square lattice.

First we make a superposition of a triangular and a hexagonal lattice as shown in figure 1. On each site of these lattices we define spins variables σ that assume the values $0, 1, \ldots, q-1$. With the prescription given in figure 2, we associate to the two decoupled lattices a q^3 vertex model on a Kagomé lattice. Note that this Kagomé lattice has three types of vertices. If the spin model satisfies the star–triangle relation then the vertex model obeys the factorisation equations and its partition function is Z invariant (Baxter 1978) so that we can write

$$-\beta F_{\mathrm{K}} = -\tfrac{1}{3} \sum_{i=1}^{3} \beta F_i \tag{1}$$

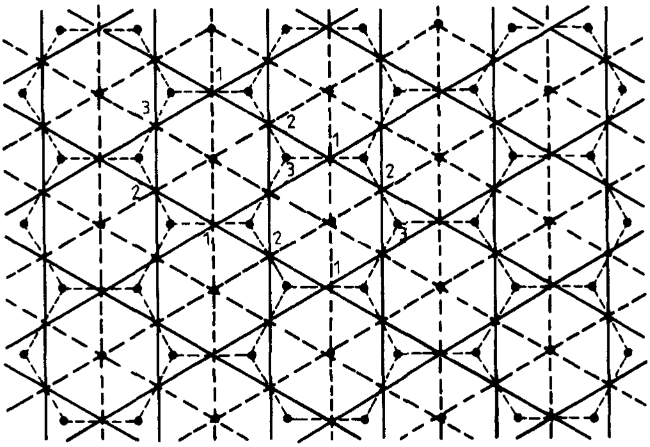

Figure 1. – – –, hexagonal lattice; ——— —, triangular lattice, ——, three types of vertices on a Kagomé lattice.

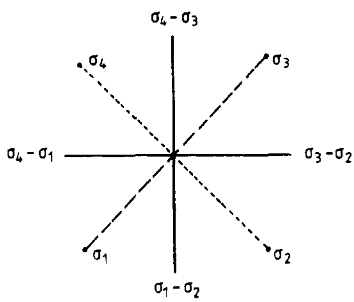

Figure 2. Vertex prescription. The difference $\sigma_i - \sigma_j$ is modulo q.

where F_K is the free energy per vertex of the Kagomé lattice and F_i is the free energy per vertex of type i of the square lattice.

It is easy to see that a Kagomé lattice with N vertices corresponds to a triangular lattice with $N/3$ sites and a hexagonal lattice with $2N/3$ sites (as we are interested in the thermodynamic limit the boundary conditions on these lattices are irrelevant); then

$$-\beta F_K = -\tfrac{1}{3}\beta F_t - \tfrac{2}{3}\beta F_h \qquad (2)$$

where F_t and F_h are the free energy per spin of the triangular and hexagonal lattices respectively.

Following Pokrowsky and Bashilov (1982) we write the star–triangle relation as functional equations (see figure 3)

$$\sum_{\sigma=0}^{q-1} K_{\sigma_1\sigma}(\theta_1)K_{\sigma_2\sigma}(\theta_2)K_{\sigma_3\sigma}(\theta_3)$$
$$= \lambda(\theta_1, \theta_2, \theta_3)K_{\sigma_2\sigma_3}(\pi - \theta_1)K_{\sigma_3\sigma_1}(\pi - \theta_2)K_{\sigma_1\sigma_2}(\pi - \theta_3) \qquad (3)$$

where $K_{\sigma\sigma'}(\theta_i)$ and $K_{\sigma\sigma'}(\pi - \theta_i)$ are the Boltzmann weights on links of type i of the hexagonal and triangular lattices respectively, $\lambda(\theta_1, \theta_2, \theta_3)$ is some symmetric function of the rapidities θ_i and $\theta_1 + \theta_2 + \theta_3 = \pi$.

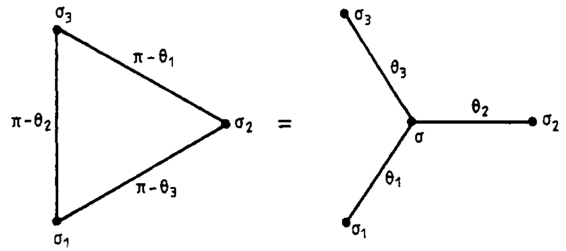

Figure 3. Star–triangle relation.

Making a spin decimation on the hexagonal lattice through the star–triangle relation we get

$$-\beta F_{\mathrm{h}} = \tfrac{1}{2} \ln \lambda - \tfrac{1}{2}\beta F_{\mathrm{t}}. \tag{4}$$

Combining the expressions (1), (2) and (4) we obtain

$$-\beta F_{\mathrm{t}} = -\tfrac{1}{2} \sum_{i=1}^{3} \beta F_i - \tfrac{1}{2} \ln \lambda. \tag{5}$$

Equation (5) is the key expression of our letter. Once the model has been solved on the square lattice and obeys the star–triangle relation we can get the result for the triangular and hexagonal lattices.

We now derive the free energy of the critical Potts model using expression (5). The Boltzmann weights for this model are

$$K_{\sigma\sigma'}(\theta_i) = 1 + v(\theta_i)\delta_{\sigma\sigma'} \tag{6}$$

where $v(\theta_i)$ is given by (see Pokrowsky and Bashilov 1982)

$$v(\theta_i) = 2\cosh(\mu\pi)\sinh[\mu(\pi-\theta_i)]/\sinh(\mu\theta_i) \tag{7}$$

and $q^{1/2} = 2\cosh\mu\pi$.

The function λ is derived from the star–triangle relation and we give its expression below:

$$\lambda(\theta_1, \theta_2, \theta_3) = \sqrt{q} \prod_{i=1}^{3} \frac{\sinh[\mu(\pi-\theta_i)]}{\sinh(\mu\theta_i)}. \tag{8}$$

The free energy per vertex of type i is given by

$$-\beta F_i = \tfrac{1}{2}\ln q + \int_{-\infty}^{\infty} \frac{\sinh[\pi x(1-\gamma)]\tanh(\pi\gamma x)\cosh[\gamma x(\pi - 2\theta_i)]}{x\sinh(\pi x)}\,\mathrm{d}x \tag{9a}$$

$$-\beta F_i = \tfrac{1}{2}\ln 4 + \ln\left(\frac{\theta_i(\pi-\theta_i)}{(2\pi)^2}\right) + 2\ln\left(\frac{\Gamma(\theta_i/2\pi)\Gamma(\frac{1}{2}-\theta_i/2\pi)}{\Gamma(\frac{1}{2}+\theta_i/2\pi)\Gamma(1-\theta_i/2\pi)}\right) \tag{9b}$$

$$-\beta F_i = \tfrac{1}{2}\ln q + \mu\pi + 2\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\mu\pi n}}{n}\tanh(\mu\pi n)\cosh[\mu n(\pi - 2\theta_i)]. \tag{9c}$$

Equations (9a), (9b) and (9c) are the free energy for the cases $q<4$, $q=4$, $q>4$ respectively and $\gamma = -\mathrm{i}\mu$.

From expressions (5), (8) and (9) we can derive the critical Potts model on a triangular lattice:

$$-\beta F_t = \tfrac{1}{2}\ln q + \tfrac{1}{2}\sum_{i=1}^{3}\int_{-\infty}^{\infty}\frac{\sinh[\pi x(1-\gamma)]\sinh(2x\gamma\theta_i)}{x\sinh(\pi x)\cosh(\pi\gamma x)}\,\mathrm{d}x \qquad (10a)$$

$$-\beta F_t = \tfrac{1}{2}\ln 4 + \sum_{i=1}^{3}\left[\ln\left(\frac{\theta_i}{2\pi}\right) + \ln\left(\frac{\Gamma(\theta_i/2\pi)\Gamma(\frac{1}{2}-\theta_i/2\pi)}{\Gamma(\frac{1}{2}+\theta_i/2\pi)\Gamma(1-\theta_i/2\pi)}\right)\right] \qquad (10b)$$

$$-\beta F_t = \tfrac{1}{2}\ln q + \mu\pi + \sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{e^{-\mu\pi n}\sinh(2\mu n\theta_i)}{n\cosh(\mu\pi n)}. \qquad (10c)$$

Here equations (10*a*), (10*b*) and (10*c*) correspond again to the cases $q<4$, $q=4$ and $q>4$ respectively. The connection with Baxter's parameters (Baxter *et al* 1978) is $\lambda=\mu\pi$, $\alpha_j=\mu(\pi-\theta_j)$ and $\tau_j=(\pi-\theta_j)/2\pi$.

Recently some new results for self-dual $Z(N)$ spin models were obtained on the square lattice (Fateev and Zamolodchikov 1982). The model is defined by the statistical weights determining the interaction between two spins σ and σ' on the neighbouring sites of the lattice. They found the following parametrisation:

$$K_n(\theta_i) = \prod_{k=0}^{n-1}\frac{\sin(\pi K/N+\theta_i/2N)}{\sin[(\pi/N)(k+1)-\theta_i/2N]} \qquad (11)$$

where N is the number of spin states, $n=\sigma-\sigma'$ (modulo N), and θ_i are the rapidities with $\theta_1+\theta_2+\theta_3=\pi$.

The functions $K_n(\theta_1)$ have the following properties: $K_n=K_{-n}=K_{N-n}$ and K_0 is chosen to be equal to one. The free energy per vertex of type i is given below:

$$-\beta F_i = \int_0^{\infty}\frac{\sinh(\frac{1}{2}x\theta_i)\sinh[\frac{1}{2}x(\pi-\theta_i)]\sinh[\frac{1}{2}\pi x(N-1)]}{x\cosh^2(\frac{1}{2}\pi x)\cosh(\frac{1}{2}N\pi x)}\,\mathrm{d}x. \qquad (12)$$

Making use of the star–triangle relation (3) we can find the function λ:

$$\lambda(\theta_1,\theta_2,\theta_3) = \sqrt{N}\prod_{i=1}^{3}\prod_{k=1}^{[N/2]}\frac{\sin[(\pi/N)(k-\frac{1}{2})+\theta_i/2N]}{\sin[\pi k/N-\theta_i/2N]} \qquad (13)$$

where $[N/2]$ means the largest integer smaller than $N/2$. Finally using the relations (5), (12) and (13) we get the self-dual solutions on a triangular lattice:

$$-\beta F_t = \tfrac{1}{2}\ln N - \tfrac{1}{2}\sum_{i=1}^{3}\int_0^{\infty}\frac{\sinh[\frac{1}{2}\pi x(N-1)]\sinh(x\theta_i)}{x\cosh(\frac{1}{2}\pi Nx)\cosh(\frac{1}{2}\pi x)\sinh(\pi x)}\,\mathrm{d}x. \qquad (14)$$

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